4) Linear D.E. (Integration Factor Method)

4. Integrating factor method

Consider an ordinary differential equation (o.d.e.) that we wish to solve to find out how the variable z depends on the variable x.

If the equation is **first order** then the highest derivative involved is a first derivative.

If it is also a **linear** equation then this means that each term can involve z either as the derivative $\frac{dz}{dx}$ OR through a single factor of z.

Any such linear first order o.d.e. can be re-arranged to give the following standard form:

$$\boxed{\frac{dz}{dx} + P_1(x)z = Q_1(x)}$$

where $P_1(x)$ and $Q_1(x)$ are functions of x, and in some cases may be constants.

A linear first order o.d.e. can be solved using the **integrating factor method**.

After writing the equation in standard form, $P_1(x)$ can be identified. One then multiplies the equation by the following "integrating factor":

$$\text{IF} = e^{\int P_1(x)dx}$$

This factor is defined so that the equation becomes equivalent to:

$$\frac{d}{dx}(\text{IF } z) = \text{IF } Q_1(x),$$

whereby integrating both sides with respect to x, gives:

$$\text{IF } z = \int \text{IF } Q_1(x)\, dx$$

Finally, division by the integrating factor (IF) gives z explicitly in terms of x, i.e. gives the solution to the equation.

مدرس المادة

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lec. 3

differential equations

في حالة كون $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ فأن المعادلة التفاضلية ستكون ليست من نوع المضبوطة (not exact)، لذلك سوف يتم استخدام معامل التكامل (IF) والذي سوف يجعل $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$$IF = e^{\int \mu_x dx} \quad \text{or} \quad e^{\int \mu_y dy} \longrightarrow$$ حسب نوع الدالة.

$$\boxed{\mu_x = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] dx}$$ function of (x).

$$\boxed{\mu_y = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dy}$$ function of (y).

EX1: solve the following D.E. :

$$y\,dx + 3 + 3x - y\,dy = 0$$

Sol:- $M dx + N dy = 0$

$$M = y \longrightarrow \frac{\partial M}{\partial y} = 1$$

$$N = 3 + 3x - y \longrightarrow \frac{\partial N}{\partial x} = 3$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ not exact

$$\mu_x = \frac{1}{3+3x-y}[1-3] \longrightarrow \mu_x = \frac{-2}{3+3x-y} = f(x,y)$$

$$\mu_y = \frac{1}{y}[3-1] \longrightarrow \mu_y = \frac{2}{y} = f(y)$$

نلاحظ ان ($\mu_x = \frac{-2}{3+3x-y}$) هي دالة لـ y و X لذلك

نهمل ونستخدم فقط ($\mu_y = \frac{2}{y}$) لانها دالة لـ (y) فقط

$$\therefore IF = e^{\int f(y)\,dy} = e^{\int \frac{2}{y}dy} = e^{2\ln y} = y^2$$

$$\boxed{IF = y^2} \longrightarrow$$ نضرب بطرفي المعادلة

$$y^3\,dx + (3y^2 + 3xy^2 - y^3)\,dy = 0$$

$$\frac{\partial M}{\partial y} = 3y^2 \quad , \quad \frac{\partial N}{\partial x} = 3y^2$$

$\therefore$ exact

$$f = \int M\,dx + g(y) \quad , \int M\,dx = \int y^3\,dx = y^3 x$$

$$f = y^3 x + g(y) \quad , \quad \frac{\partial f}{\partial y} = 3xy^2 + g'(y) = N$$

$$\cancel{3xy^2} + g'(y) = 3y^2 + \cancel{3xy^2} - y^3$$

$$\therefore g'(y) = 3y^2 - y^3$$

$$g(y) = \int g'(y) = \int 3y^2 - y^3\,dy$$

$$g(y) = y^3 - \frac{1}{4}y^4 + c$$

$$\therefore f = y^3 x + y^3 - \frac{1}{4}y^4 + c.$$

4- linear Differential equations:

general form of linear D.E. is:

$$\boxed{\frac{dy}{dx} + P(x) \cdot y = Q(x)} \rightarrow$$ الشكل العام للمعادلة الخطية

Solution of this equation can be obtain as following:

$$\boxed{y \cdot IF = \int Q(x) \cdot IF \, dx + C} \rightarrow$$ قانون حل المعادلة التفاضلية الخطية.

Ex1: solve: $\frac{dy}{dx} + \frac{y}{x} = 1$

Sol:-

$$P(x) = \frac{1}{x}$$

$$Q(x) = 1$$

$$IF = e^{\int P(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

$$\therefore y \cdot IF = \int Q(x) \cdot IF \, dx + C$$

$$y \cdot x = \int 1 \cdot x \, dx + C$$

$$yx = \frac{x^2}{2} + C$$

Ex2: Solve : $\frac{dx}{dt} + \frac{10x}{2t+5} = 10$

Sol: $\frac{dx}{dt} + \frac{10x}{2t+5} = 10 \iff \frac{dx}{dt} + P(t)\,x = Q(t)$

$\therefore\ P(t) = \frac{10}{2t+5}$

$Q(t) = 10$

$$IF = e^{\int P(t)dt} = e^{\int \frac{10}{2t+5}dt}$$

مشتقة المقام = 2
لذلك نضرب البسط $\times \frac{1}{5}$
والتكامل $(*5)$
(اي نقسم على 5 ونضرب $*5$)

$$I.F. = e^{5\ln(2t+5)} = e^{\ln(2t+5)^5} = e^{5\int\frac{\frac{1}{5}\cdot 10\,dt}{2t+5}}$$

$$\therefore\ IF = (2t+5)^5$$

$$X \cdot IF = \int Q(t) \cdot IF\,dt + C$$

$$X \cdot (2t+5)^5 = \int 10 \cdot (2t+5)^5\,dt + C$$

$$X \cdot (2t+5)^5 = 5\int \frac{10}{5} \cdot (2t+5)^5\,dt + C$$

نوفر مشتقة داخل القوس والتي تساوي (2) لذلك نقسم ١٠ على 5 ونضرب التكامل $(*5)$

$$X \cdot (2t+5)^5 = \frac{5}{6}(2t+5)^6 + C$$

EX3: $\frac{dy}{dx} + 5y = 50$

$\Leftrightarrow \frac{dy}{dx} + P(x)\,y = Q(x)$

$\therefore P(x) = 5$

$Q(x) = 50$

$IF = e^{\int P(x)dx} = e^{\int 5dx} = e^{5x}$

$y \cdot IF = \int Q(x) \cdot IF\, dx + c$

$y \cdot e^{5x} = \int 50 \cdot e^{5x}\, dx + c$

مشتقة الأس = 5

$\therefore \int e^{5x} = \frac{1}{5}\int 5 \cdot e^{5x} dx$

$= \frac{1}{5} e^{5x} + c$

$y\, e^{5x} = \frac{50}{5} e^{5x} + c$

$y\, e^{5x} = 10\, e^{5x} + c$

أي إن:

$\int e^{au} = \frac{1}{a} e^{au} + c$